

ON THE HAUSDORFF DIMENSION OF SOME SETS OF NUMBERS DEFINED THROUGH THE DIGITS OF THEIR Q -CANTOR SERIES EXPANSIONS

DYLAN AIREY AND BILL MANCE

ABSTRACT. Following in the footsteps of P. Erdős, A. Rényi, and T. Šalát we compute the Hausdorff dimension of sets of numbers whose digits with respect to their Q -Cantor series expansions satisfy various statistical properties. In particular, we consider difference sets associated with various notions of normality and sets of numbers with a prescribed range of digits.

1. INTRODUCTION

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by P. Erdős and A. Rényi in [4] and [5] and by A. Rényi in [14], [15], and [16] and by P. Turán in [19].

The Q -Cantor series expansions, first studied by G. Cantor in [3], are a natural generalization of the b -ary expansions.¹ Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^\mathbb{N}$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_n)_{n=1}^\infty$, the Q -Cantor series expansion of a real number x is the (unique)² expansion of the form

$$(1.1) \quad x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \dots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.1) with the notation $x = E_0.E_1E_2E_3 \dots$ w.r.t. Q .

A *block* is an ordered tuple of non-negative integers, a *block of length k* is an ordered k -tuple of integers, and *block of length k in base b* is an ordered k -tuple of integers in $\{0, 1, \dots, b - 1\}$.

Let

$$Q_n^{(k)} := \sum_{j=1}^n \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}} \text{ and } T_{Q,n}(x) := \left(\prod_{j=1}^n q_j \right) x \pmod{1}.$$

A. Rényi [15] defined a real number x to be *normal* with respect to Q if for all blocks B of length 1,

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1.$$

Research of the authors is partially supported by the U.S. NSF grant DMS-0943870.

¹G. Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [7].

²Uniqueness can be proven in the same way as for the b -ary expansions.

If $q_n = b$ for all n and we restrict B to consist of only digits less than b , then (1.2) is equivalent to *simple normality in base b* , but not equivalent to *normality in base b* . A basic sequence Q is *k -divergent* if $\lim_{n \rightarrow \infty} Q_n^{(k)} = \infty$, *fully divergent* if Q is k -divergent for all k , and *k -convergent* if it is not k -divergent. A basic sequence Q is *infinite in limit* if $q_n \rightarrow \infty$.

Definition 1.1. A real number x is *Q -normal of order k* if for all blocks B of length k ,

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1.$$

We let $\mathcal{N}_k(Q)$ be the set of numbers that are Q -normal of order k . The real number x is *Q -normal* if $x \in \mathcal{N}(Q) := \bigcap_{k=1}^{\infty} \mathcal{N}_k(Q)$. x is *Q -ratio normal of order k* (here we write $x \in \mathcal{RN}_k(Q)$) if for all blocks B_1 and B_2 of length k

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B_1, x)}{N_n^Q(B_2, x)} = 1.$$

x is *Q -ratio normal* if $x \in \mathcal{RN}(Q) := \bigcap_{k=1}^{\infty} \mathcal{RN}_k(Q)$. A real number x is *Q -distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathcal{DN}(Q)$ be the set of Q -distribution normal numbers.

It was proven in [11] that the directed graph in Figure 1 gives the complete containment relationships between these notions when Q is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$. The set labeled on vertex A is a subset of the set labeled on vertex B if and only if there is a directed path from A to B . For example, $\mathcal{N}(Q) \cap \mathcal{DN}(Q) \subseteq \mathcal{RN}(Q)$, so all numbers that are Q -normal and Q -distribution normal are also Q -ratio normal.

Note that in base b , where $q_n = b$ for all n , the corresponding notions of Q -normality, Q -ratio normality, and Q -distribution normality are equivalent. This equivalence is fundamental in the study of normality in base b .

It follows from a well known result of H. Weyl [24, 25] that $\mathcal{DN}(Q)$ is a set of full Lebesgue measure for every basic sequence Q . We will need the following result of the second author [13] later in this paper.

Theorem 1.2. ³ Suppose that Q that is infinite in limit. Then $\mathcal{N}_k(Q)$ and $\mathcal{RN}_k(Q)$ are of full measure if and only if Q is k -divergent. The sets $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ are of full measure if and only if Q is fully divergent.

Based on Figure 1 and Theorem 1.2 it is natural to ask for the Hausdorff dimension of the difference sets. It was proven in [12] that for every basic sequence Q that is infinite in limit

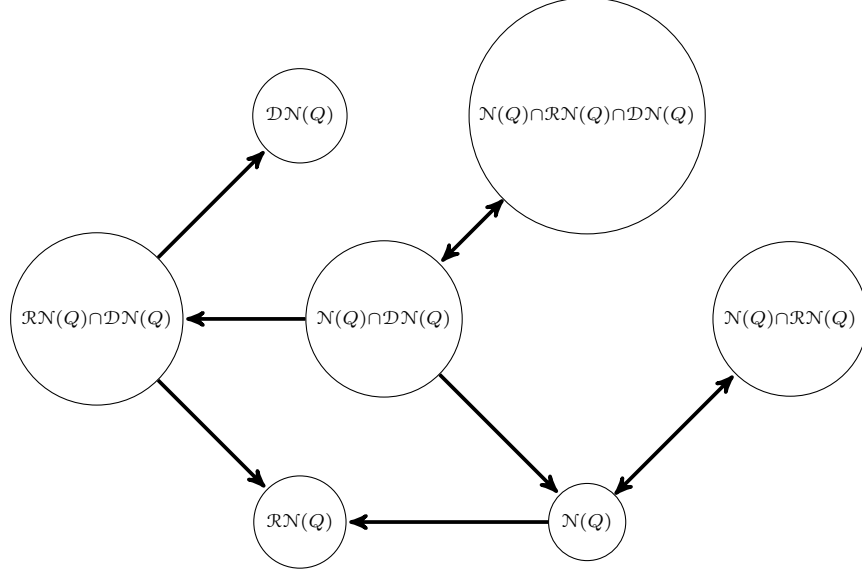
$$\dim_{\text{H}}(\mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = \dim_{\text{H}}(\mathcal{DN}(Q) \setminus \mathcal{RN}(Q)) = 1.$$

Using different methods we will prove the following theorem.

Theorem 1.3. Every non-empty difference set expressed in terms of $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$, possibly involving intersections and unions, has full Hausdorff dimension for every Q that is infinite in limit, except for the set $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$,

³Early work in this direction has been done by A. Rényi [15], T. Šalát [22], and F. Schweiger [18].

FIGURE 1.



It will be shown that the set $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ has full Hausdorff dimension for a more restricted class of basic sequences in Theorem 3.4. We should note that we can not hope to establish $\dim_{\mathbb{H}}(\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = 1$ for all Q that are infinite in limit. This follows from the result in [13] that $\mathcal{N}(Q) = \emptyset$ when Q is infinite in limit and not fully divergent.

A surprising property of Q -normality of order k is that we may not conclude that $\mathcal{N}_k(Q) \subseteq \mathcal{N}_j(Q)$ for all $j < k$ like we may for the b -ary expansions. In fact, it was shown in [10] that for every k there exists a basic sequence Q and a real number x such that $\mathcal{N}_k(Q) \setminus \bigcup_{j=1}^{k-1} \mathcal{N}_j(Q)$ is non-empty. Thus, we will have to be more careful in stating exactly what our theorems prove since lack of Q -normality of order 2 does not imply lack of Q -normality of order 338, for example. Furthermore, we will greatly expand on this result in Theorem 3.5 where for each natural number ℓ we exhibit a class of basic sequences such that

$$\dim_{\mathbb{H}} \left(\bigcap_{j=\ell}^{\infty} \mathcal{N}_j(Q) \setminus \bigcup_{j=1}^{\ell-1} \mathcal{N}_j(Q) \right) = 1.$$

For $x = E_0.E_1E_2 \dots$ w.r.t. Q , define the set

$$\mathcal{S}_Q(x) = \{E_1, E_2, E_3, \dots\}.$$

P. Erdős and A. Rényi [4] proved the following theorems.

Theorem 1.4 (P. Erdős and A. Rényi). *If Q is 1-convergent, then $\mathcal{S}_Q(x)$ has density 0 for almost every real number x .*

Theorem 1.5 (P. Erdős and A. Rényi). *For $x = E_0.E_1E_2 \dots$ w.r.t. Q , let $d_n(x)$ denote the number of different numbers in the sequence E_1, \dots, E_n . If Q is 1-convergent, then for almost every x we have $\lim_{n \rightarrow \infty} \frac{d_n(x)}{n} = 1$.*

It should be noted that T. Šalát [23] considered sets related to those mentioned in Theorem 1.4 and Theorem 1.5. We will need the following definition from [2].

Definition 1.6. For $S \subseteq \mathbb{Z}$, define the *mass dimension* of S to be the limit

$$\dim_M(S) = \lim_{n \rightarrow \infty} \frac{\log \#(S \cap (-n/2, n/2))}{\log n},$$

if it exists.

We note that an *upper mass dimension* and a *lower mass dimension* may be defined similarly by changing the limit in Definition 1.6 to a \limsup or a \liminf .

For non-empty $S \subseteq \mathbb{N}_0$, define

$$\mathcal{W}_Q(S) = \{x \in \mathbb{R} : \mathcal{S}_Q(x) = S\}.$$

We will build on Theorem 1.4 and Theorem 1.5 by proving the following theorem.

Theorem 1.7. *If Q is infinite in limit, $\lim_{n \rightarrow \infty} \frac{\log q_n}{\sum_{i=1}^n \log q_i} = 0$, and $S \subseteq \mathbb{N}$ such that $\min S < \min Q$ and $\dim_M(S)$ exists, then*

$$\dim_H(\mathcal{W}_Q(S)) = \dim_M(S).$$

T. Šalát proved in [21] that under some conditions on the basic sequence Q the set of real numbers whose digits in their Q -Cantor expansion is bounded has zero Hausdorff dimension. We remark that his result may be sharpened with his conditions weakened by use of our Lemma 2.4 instead of Satz 1 from [20]. The proof of this otherwise follows identically to his original proof, so we do not record it in this paper.

If Q is infinite in limit and not fully divergent, then $\lambda(\mathcal{RN}(Q)) = 0$. We will show as a consequence of the following theorem that $\dim_H(\mathcal{RN}(Q)) = 1$ whenever Q is infinite in limit.

Theorem 1.8. *If Q is infinite in limit, then $\dim_H(\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = 1$.*

Lastly, we remark that some of the techniques developed in this paper and Lemma 2.4 are used to study fractals associated with normality preserving operations in [1].

2. LEMMAS

Let (n_k) be a sequence of positive integers and (c_k) be a sequence of positive numbers such that $n_k \geq 2$, $0 < c_k < 1$, $n_1 c_1 \leq \delta$, and $n_k c_k \leq 1$, where δ is a positive real number. For any k , let $D_k = \{(i_1, \dots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$, and $D = \bigcup D_k$, where $D_0 = \emptyset$. If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, $\tau = (\tau_1, \dots, \tau_m) \in D_m$, put $\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m)$.

Definition 2.1. Suppose J is a closed interval of length δ . The collection of closed subintervals $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ of J has *homogeneous Moran structure* if:

- (1) $J_\emptyset = J$;
- (2) $\forall k \geq 0, \sigma \in D_k, J_{\sigma * 1}, \dots, J_{\sigma * n_{k+1}}$ are subintervals of J_σ and $\mathring{J}_{\sigma * i} \cap \mathring{J}_{\sigma * j} = \emptyset$ for $i \neq j$;
- (3) $\forall k \geq 1, \forall \sigma \in D_{k-1}, 1 \leq j \leq n_k, c_k = \frac{\lambda(J_{\sigma * j})}{\lambda(J_\sigma)}$.

Suppose that \mathcal{F} is a collection of closed subintervals of J having homogeneous Moran structure. Let $E(\mathcal{F}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma$. We say $E(\mathcal{F})$ is a *homogeneous Moran set determined by \mathcal{F}* , or it is a *homogeneous Moran set determined by $J, (n_k), (c_k)$* . We will need the following theorem of D. Feng, Z. Wen, and J. Wu from [6].

Theorem 2.2 (D. Feng, Z. Wen, and J. Wu). *If S is a homogeneous Moran set determined by $J, (n_k), (c_k)$, then*

$$\liminf_{k \rightarrow \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}} \leq \dim_H(S) \leq \liminf_{k \rightarrow \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}.$$

Given basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of non-negative integers $s = (s_i), t = (t_i), v = (v_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_i)$ such that $I_i \subseteq \{0, 1, \dots, \beta_i - 1\}$, define the set $\Theta(\alpha, \beta, s, t, v, F, I)$ as follows. Let $Q = Q(\alpha, \beta, s, t, v) = (q_n)$ be the following basic sequence:

$$(2.1) \quad [[\alpha_1]^{s_1} [\beta_1]^{t_1}]^{v_1} [[\alpha_2]^{s_2} [\beta_2]^{t_2}]^{v_2} [[\alpha_3]^{s_3} [\beta_3]^{t_3}]^{v_3} \cdots.$$

Define the function

$$i(n) = \min \left\{ t : \sum_{i=1}^{t-1} v_i(s_i + t_i) < n \right\}.$$

Set

$$\Phi_\alpha(i, c, d) = \sum_{j=1}^{i-1} v_j s_j + c s_i + d$$

where $0 \leq c < v_i$ and $0 \leq d < s_i$ and let the functions $i_\alpha(n)$, $c_\alpha(n)$, and $d_\alpha(n)$ be such that $\Phi_\alpha^{-1}(n) = (i_\alpha(n), c_\alpha(n), d_\alpha(n))$. Note this is possible since Φ_α is a bijection from $\mathcal{U} = \{(i, c, d) \in \mathbb{N}^3 : 0 \leq c < v_i, 0 \leq d < s_i\}$ to \mathbb{N} . Define the functions

$$G(n) = \sum_{j=1}^{i_\alpha(n)-1} v_j(s_j + t_j) + c_\alpha(n)(s_{i_\alpha(n)} + t_{i_\alpha(n)}) + d_\alpha(n)$$

and $g(n) = \min \{t : G(t) \geq n\}$. Note that $i_\alpha(g(n)) = i(n)$ and $c_\alpha(g(n)) = c_\alpha(n)$. Furthermore, define $C_\alpha(n) = \left(\sum_{j=1}^{i_\alpha(n)-1} u_j \right) + c_\alpha(n)$.

We consider the condition on n

$$(2.2) \quad \left(n - \sum_{j=1}^{i(n)-1} v_j(s_j + t_j) \right) \bmod (s_{i(n)} + t_{i(n)}) \geq s_{i(n)}.$$

Define the intervals

$$V(n) = \begin{cases} I_{i(n)} & \text{if condition (2.2) holds} \\ [F_{G(n)}, F_{G(n)} + 1) & \text{else} \end{cases}.$$

That is, we choose digits from $I_{i(n)}$ in positions corresponding to the bases obtained from the sequence β and choose a specific digit from F for the bases obtained from the sequence α . Set

$$\Theta(\alpha, \beta, s, t, v, F, I) = \{x = 0.E_1 E_2 \cdots \text{ w.r.t. } Q : E_n \in V(n)\}.$$

We will need the following basic lemma to prove Lemma 2.4 and elsewhere in this paper.

Lemma 2.3. *Let L be a real number and $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of positive real numbers such that*

$$\sum_{n=1}^\infty b_n = \infty \text{ and } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L.$$

Then

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = L.$$

Lemma 2.4. *Given basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of non-negative integers $s = (s_i), t = (t_i), v = (v_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_i)$ such that $I_i \subseteq \{0, 1, \dots, \beta_i - 1\}$ such that the following conditions hold:*

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{s_n \log \alpha_n}{\sum_{i=1}^{n-1} v_i t_i \log \beta_i} = 0;$$

$$(2.4) \quad \lim_{n \rightarrow \infty} \frac{s_n \log \alpha_n}{t_n \log \beta_n} = 0.$$

Then

$$\dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) = \gamma := \lim_{n \rightarrow \infty} \frac{\log |I_n|}{\log \beta_n}.$$

Proof. Note that $\Theta(\alpha, \beta, s, t, v, F, I)$ is a homogeneous Moran set with

$$n_k = \begin{cases} |I_k| & \text{if } q_k = \beta_{i(k)} \\ 1 & \text{if } q_k = \alpha_{i(k)} \end{cases}$$

and $c_k = \frac{1}{q_k}$. Thus

$$\begin{aligned} \dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) &\geq \liminf_{k \rightarrow \infty} \frac{\log n_1 n_2 \dots n_k}{-\log c_1 c_2 \dots c_{k+1} n_{k+1}} \\ &\geq \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{i(n)-1} \sum_{k=1}^{u_j} t_j \log |I_j| + \sum_{j=1}^{b(n)} t_{i(n)} \log |I_{i(n)}|}{\sum_{j=1}^{i(n)-1} \sum_{k=1}^{u_j} [t_j \log \beta_j + s_j \log \alpha_j] + \sum_{j=1}^{b(n)} [t_{i(n)} \log \beta_{i(n)} + s_{i(n)} \log \alpha_{i(n)}] + s_{i(n)} \log \alpha_{i(n)}} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\sum_{j=1}^{i(n)-1} u_j t_j \gamma \log \beta_j \right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\sum_{j=1}^{i(n)-1} \sum_{k=1}^{u_j} [t_j \log \beta_j + s_j \log \alpha_j] + \sum_{j=1}^{b(n)} [t_{i(n)} \log \beta_{i(n)} + s_{i(n)} \log \alpha_{i(n)}] + s_{i(n)} \log \alpha_{i(n)}} \end{aligned}$$

where we have used Lemma 2.3.

$$= \lim_{n \rightarrow \infty} \frac{\left(\sum_{j=1}^{i(n)-1} u_j t_j \gamma \log \beta_j \right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\left(\sum_{j=1}^{i(n)-1} u_j t_j \log \beta_j \right) + b(n) t_{i(n)} \log \beta_{i(n)} + s_{i(n)} \log \alpha_{i(n)}}$$

which follows from (2.4).

$$= \lim_{n \rightarrow \infty} \frac{\left(\sum_{j=1}^{i(n)-1} u_j t_j \gamma \log \beta_j \right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\left(\sum_{j=1}^{i(n)-1} u_j t_j \log \beta_j \right) + b(n) t_{i(n)} \log \beta_{i(n)}} = \gamma.$$

which we get from (2.3). The upper bound follows from a similar calculation. \square

For a sequence of real numbers $X = (x_n)$ with $x_n \in [0, 1]$ and an interval $I \subseteq [0, 1]$, define $A_n(I, X) = \#\{i \leq n : x_i \in I\}$. We will need the following standard definition and lemma that we quote from [8].

Definition 2.5. Let $X = (x_1, \dots, x_N)$ be a finite sequence of real numbers. The number

$$D_N = D_N(X) = \sup_{0 \leq \alpha \leq \beta \leq 1} \left| \frac{A_N([\alpha, \beta), X)}{N} - (\beta - \alpha) \right|$$

is called the *discrepancy* of the sequence ω .

It is well known that a sequence X is uniformly distributed mod 1 if and only if $D_N(X) \rightarrow 0$.

Lemma 2.6. Let x_1, x_2, \dots, x_N and y_1, y_2, \dots, y_N be two finite sequences in $[0, 1)$. Suppose $\epsilon_1, \epsilon_2, \dots, \epsilon_N$ are non-negative numbers such that $|x_n - y_n| \leq \epsilon_n$ for $1 \leq n \leq N$. Then, for any $\epsilon \geq 0$, we have

$$|D_N(x_1, \dots, x_N) - D_N(y_1, \dots, y_N)| \leq 2\epsilon + \frac{\overline{N}(\epsilon)}{N},$$

where $\overline{N}(\epsilon)$ denotes the number of n , $1 \leq n \leq N$, such that $\epsilon_n > \epsilon$.

3. RESULTS

We will compute the Hausdorff dimension of difference sets formed by taking unions or intersections of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$. Every other similar result will follow as a corollary of one of these theorems, by using similar techniques, or by Figure 1.

Proof of Theorem 1.8. Let $P = (p_i)$ with $p_i = \lfloor \log i \rfloor + 2$ and $\xi \in \mathcal{N}(P)$ with $\xi = .F_1 F_2 \dots$ w.r.t. P . Fix a sequence $X = (x_n)$ that is uniformly distributed modulo 1. Define the sequences

$$\nu_n = \min \left\{ t : \frac{\sum_{i=0}^{n-1} \log q_{I(n-1)+i}}{\sum_{i=0}^{j-I(n-1)-1} \log q_{I(n-1)+i}} < \frac{1}{n}, \forall j \geq t \right\};$$

$$v_{n,k} = \min \left\{ t : \frac{Q_n^{(k)}}{\sum_{i=1}^j P_{i-k+1}^{(k)}} < \frac{1}{n}, \forall j \geq t \right\};$$

$$L_0 = 0;$$

$$L_n = \max \left\{ \min \{ t : \log(q_j) > n, \forall j \geq t \}, L_{n-1} + n^2, L_{n-1} + \nu_n, \max_{k \leq n} \{ v_{n,k} \} \right\}$$

and set $i(n) = \max \{ j : L_j \leq n \}$. Note that ν_n and $v_{n,k}$ exist since Q is infinite in limit and P is fully divergent. Define the set

$$S = \bigcup_{n=1}^{\infty} \{L_n, L_n + 1, \dots, L_n + n - 1\}.$$

Note that this set has density 0 since

$$\frac{\sum_{i=1}^n i}{\sum_{i=1}^n i + t_i} \leq \frac{\sum_{i=1}^n i}{\sum_{i=1}^n i + i^2} \rightarrow 0 \text{ as } n \text{ goes to infinity.}$$

Define the intervals

$$V(n) = \begin{cases} [F_{n-L_i}, F_{n-L_i} + 1) & \text{if } n \in [L_i, L_i + 1, \dots, L_i + i] \\ [x_n q_n - \omega_n, x_n q_n + \omega_n) \cap [\lfloor \log i(n) \rfloor, q_n - 1] & \text{else} \end{cases}$$

where

$$\omega_n = q_n^{1-\epsilon_i} \text{ and } \epsilon_i = \frac{\min \{\log q_1 \cdots q_{i-1}, \log q_i\}^{1/2}}{\log q_i}$$

Set $\Lambda_Q = \{x = .E_1 E_2 \text{ w.r.t. } Q : E_n \in V(n)\}$. We claim that $\Lambda_Q \subseteq \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ and $\dim_{\mathbf{H}}(\Lambda_Q) = 1$. Let $x \in \Lambda_Q$ and let B be a block of length k . Note that by the definition of L_n , there are only finitely many values $n \in \mathbb{N} \setminus S$ such that B occurs at position n in the Q -Cantor series expansion of x . This is because all digits E_n with $n \in \mathbb{N} \setminus S$ must be greater than $\lceil \log i(n) \rceil$ by the definition of $V(n)$ and since $i(n)$ tends to infinity as n does. Thus, if m is the maximum digit for the block B , we have that for $n \in \mathbb{N} \setminus S$ with $i(n) > m$, that $E_n > m$. Thus $N_n^Q(B, x) = \sum_{i=1}^{i(n)} N_{i-k+1}^P(B, \xi) + O(1)$. So for any two blocks B_1 and B_2 of length k , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{N_n^Q(B_1, x)}{N_n^Q(B_2, x)} &= \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_1, \xi) + O(1)}{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_2, \xi) + O(1)} \\ &= \lim_{n \rightarrow \infty} \frac{N_{n-k+1}^P(B_1, \xi)}{N_{n-k+1}^P(B_2, \xi)} = 1. \end{aligned}$$

Thus $x \in \mathcal{RN}(Q)$.

Consider the sequence $Y = \left(\frac{E_n}{q_n}\right)$. For $n \in \mathbb{N} \setminus S$, we have $\left|\frac{E_n}{q_n} - x_n\right| < \frac{\omega_n}{q_n}$, which tends to 0 as n goes to infinity. We therefore have for $\epsilon > 0$ that $\overline{N}(\epsilon) = O(1) + \#S \cap \{1, \dots, N\}$. Thus by Lemma 2.6

$$|D_N(X) - D_N(Y)| < 2\epsilon + \frac{O(1)}{N} + \frac{\#S \cap \{1, \dots, N\}}{N} \rightarrow 2\epsilon$$

as N tends to infinity. Since the inequality holds for all $\epsilon > 0$, we have that $\left(\frac{E_n}{q_n}\right)$ is uniformly distributed mod 1. Thus $x \in \mathcal{DN}(Q)$.

Note that

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{\sum_{i=1}^{i(n)} P_{i-k+1}^{(k)}} = 1.$$

However,

$$\lim_{n \rightarrow \infty} \frac{Q_n^{(k)}}{\sum_{i=1}^{i(n)} P_{i-k+1}^{(k)}} = 0$$

by the definition of L_n , so $x \notin \mathcal{N}(Q)$. Thus $\Lambda_Q \subseteq \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$.

Evidently Λ_Q is a homogeneous Moran set with $n_k = |V(k)|$ and $c_k = \frac{1}{q_k}$. Thus

$$\begin{aligned} \dim_{\mathbf{H}}(\Lambda_Q) &\geq \liminf_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_{k+1} n_{k+1}} \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^k \chi_{\mathbb{N} \setminus S}(i) (1 - \epsilon_i) \log q_i}{\sum_{i=1}^k \log q_i + \log q_{k+1}} \\ &= \liminf_{n \rightarrow \infty} \left(1 - \frac{\sum_{j=1}^{i(n)} \sum_{k=0}^{j-1} \log q_{L_j+k}}{\sum_{j=1}^{i(n)} \sum_{k=0}^{i(j)-i(j-1)} \log q_{L_j+k}} \right) \\ &= \liminf_{n \rightarrow \infty} \left(1 - \frac{\sum_{i=0}^{n-1} \log q_{L_n+i}}{\sum_{i=0}^{L_n-L_{n-1}} \log q_{L_n+i}} \right) = 1 \end{aligned}$$

by the definition of L_n . Thus

$$\dim_H(\Lambda_Q) = 1 \text{ and } \dim_H(\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = 1.$$

□

Corollary 3.1. *If Q is infinite in limit, then $\dim_H(\mathcal{RN}(Q)) = 1$.*

Theorem 3.2. *If Q is infinite in limit, then*

$$\dim_H\left(\mathcal{RN}(Q) \setminus \left(\bigcup_{j=1}^{\infty} \mathcal{N}_j(Q) \cup \mathcal{DN}(Q)\right)\right) = 1.$$

Proof. The proof is the same as Theorem 1.8, but with $X = (x_n)$ a sequence that is not uniformly distributed mod 1. □

Theorem 3.3. *If Q is infinite in limit, then*

$$\dim_H\left(\mathcal{DN}(Q) \setminus \bigcup_{j=1}^{\infty} \mathcal{RN}_j(Q)\right) = 1.$$

Proof. The proof is the same as Theorem 1.8, but we choose $\xi = .E_1 E_2 \dots$ w.r.t. P such that the digit 0 never occurs. □

We will need to refer to the following four conditions.

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} = 0;$$

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} > 0;$$

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{\alpha_n^k}{s_n} = 0;$$

$$(3.4) \quad \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n v_i s_i}{\sum_{i=1}^n v_i (s_i + t_i)} = 0.$$

Theorem 3.4. *Suppose that $Q = Q(\alpha, \beta, s, t, v)$ is infinite in limit, k -divergent (resp. fully divergent), and satisfies conditions (2.3), (2.4), (3.1) for all k , (3.3), and (3.4). If $\alpha_i = o(\beta_i)$, then*

$$\dim_H\left(\bigcap_{j=1}^k \mathcal{N}_j(Q) \setminus \mathcal{DN}(Q)\right) = 1 \text{ (resp. } \dim_H(\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = 1 \text{)}.$$

Proof. We will prove the statement for when $Q = (q_n)$ is fully divergent. The proof for when Q is k -divergent follows similarly. Define the basic sequence P by

$$P = [\alpha_1]^{s_1 v_1} [\alpha_2]^{s_2 v_2} [\alpha_3]^{s_3 v_3} [\alpha_4]^{s_4 v_4} \dots$$

We note that P is fully divergent since Q is fully divergent. By Theorem 1.2, there exists a real number $\xi = E_0.E_1 E_2 \dots$ w.r.t. P that is an element of $\mathcal{N}(P)$. Set

$$I_i = \left\{ \alpha_i, \alpha_i + 1, \dots, \left\lfloor \beta_i^{1 - (1/\log \beta_i)^{1/2}} \right\rfloor + 1 \right\}$$

and $F_i = E_i$. Note that $\lim_{n \rightarrow \infty} \log |I_n| / \log \beta_n = 1$, so $\dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) = 1$ by Lemma 2.4. We now wish to show that

$$\Theta(\alpha, \beta, s, t, v, F, I) \subseteq \mathcal{N}(Q) \setminus \mathcal{DN}(Q).$$

Let k and n be natural numbers, B be a block of length k , and $x \in \Theta(\alpha, \beta, s, t, v, F, I)$. We wish to show that

$$N_{g(n)}^P(B, \xi) - kC_\alpha(g(n)) \leq N_n^Q(B, x) \leq N_{g(n)}^P(B, \xi) + O(1).$$

Let m be the maximum digit in the block B . Since $\min I_i \rightarrow \infty$, we know that there are only finitely many indices i such that $m > \min I_i$. Thus, there are at most finitely many occurrences of B starting at position n when $q_n = \beta_{i(n)}$. If every occurrence of B in ξ occurs at the corresponding place in x , then we have

$$N_{g(n)}^P(B, \xi) + O(1) = N_n^Q(B, x).$$

If some of the occurrences of B in ξ do not occur in the corresponding places in x , then we have $N_n^Q(B, x) \leq N_{g(n)}^P(B, \xi)$.

On the other hand, the total number of places up to position n where B can occur in the P -Cantor series expansion of ξ but B does not occur in the corresponding positions in the Q -Cantor series expansion of x is at most $kC_\alpha(n)$, the total length of the last k terms of the substrings $[\alpha_i]^{s_i}$ of P . Thus

$$N_{g(n)}^P(B, \xi) - kC_\alpha(g(n)) \leq N_n^Q(B, x) \leq N_{g(n)}^P(B, \xi) + O(1).$$

Many of the following calculations use Lemma 2.3. Note that

$$P_n^{(k)} = \sum_{j=1}^{i_\alpha(n)-1} \frac{s_j v_j}{\alpha_j^k} + \frac{s_{i(n)} b_\alpha(n)}{\alpha_{i(n)}^k}$$

and

$$\begin{aligned} Q_n^{(k)} &= \left(\sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k} + \left(\sum_{l=1}^{k-1} \frac{v_j}{\beta_j^l \alpha_j^{k-l}} + \frac{v_j}{\alpha_j^l \beta_j^{k-l}} \right) \right) \\ &\quad + \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k} + \left(\sum_{l=1}^{k-1} \frac{v_{i(n)}}{\beta_{i(n)}^l \alpha_{i(n)}^{k-l}} + \frac{v_j}{\alpha_{i(n)}^l \beta_{i(n)}^{k-l}} \right). \end{aligned}$$

Note that by (2.3) and (2.4), we have that

$$\lim_{n \rightarrow \infty} \frac{Q_n^{(k)}}{\left(\sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k} \right) + \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k}} = 1.$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{Q_n^{(k)}}{P_{g(n)}^{(k)}} &= \lim_{n \rightarrow \infty} \frac{\left(\sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k} \right) + \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k}}{\left(\sum_{j=1}^{i(n)-1} \frac{s_j v_j}{\alpha_j^k} \right) + \frac{c(n)s_{i(n)}}{\alpha_{i(n)}^k}} \\ &= \lim_{n \rightarrow \infty} \frac{s_n - k}{s_n} + \frac{(t_n - k)\alpha_n^k}{s_n \beta_n^k} = 1 + \lim_{n \rightarrow \infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} = 1. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{C_\alpha(g(n))}{P_{g(n)}^{(k)}} &= \lim_{n \rightarrow \infty} \frac{\left(\sum_{j=1}^{i(n)-1} v_j \right) + c(n)}{\left(\sum_{j=1}^{i(n)-1} \frac{s_j v_j - k}{\alpha_j^k} \right) + \frac{c(n)s_{i(n)} - k}{\alpha_{i(n)}^k}} \\ &= \lim_{n \rightarrow \infty} \frac{\alpha_n^k}{s_n - k/v_n} = \lim_{n \rightarrow \infty} \frac{\alpha_n^k}{s_n} = 0. \end{aligned}$$

Since $\xi \in \mathcal{N}(P)$, we have that

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = \lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{P_{g(n)}^{(k)}} = 1.$$

Therefore, $x \in \mathcal{N}(Q)$.

For n where $q_n = \beta_{i(n)}$, we have

$$(3.5) \quad \frac{E_n}{q_n} \leq \frac{\beta_{i(n)}^{1 - \log^{-1/2} \beta_{i(n)}}}{\beta_{i(n)}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Up to position n there are at least $\sum_{j=1}^{i(n)} v_i t_i + c(n) t_{i(n)}$ such places where (3.5) holds. By (3.4), we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{i(n)} v_i t_i + c(n) t_{i(n)}}{n} = 1$$

so the sequence $\left(\frac{E_n}{q_n}\right)$ is not uniformly distributed mod 1. Thus $x \notin \mathcal{DN}(Q)$ and $\Theta(\alpha, \beta, s, t, v, F, I) \subseteq \mathcal{N}(Q) \setminus \mathcal{DN}(Q)$, which implies that $\dim_{\mathbb{H}}(\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = 1$. \square

Theorem 3.5. *Suppose that $Q = Q(\alpha, \beta, s, t, v)$ is infinite in limit, fully divergent, and satisfies conditions (2.3), (2.4), (3.1) for $k \geq \ell$, (3.2) for $\ell < k$, and (3.3). Then*

$$\dim_H \left(\bigcap_{j=\ell}^{\infty} \mathcal{N}_j(Q) \setminus \bigcup_{j=1}^{\ell-1} \mathcal{N}_j(Q) \right) = 1.$$

Proof. Define the same basic sequence P and sequences I and F as in the proof of Theorem 3.4. The same arguments regarding the asymptotics of $N_n^Q(B, x)$ for $x \in \Theta(\alpha, \beta, s, t, F, I)$ hold, so

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{P_{g(n)}^{(k)}} = 1.$$

But since (3.1) holds for $k \geq \ell$, we have that

$$\lim_{n \rightarrow \infty} \frac{Q_n^{(k)}}{P_{g(n)}^{(k)}} = 1 + \lim_{n \rightarrow \infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} = 1.$$

Thus x is Q -normal of orders greater than or equal to ℓ . \square

Example 3.6. Set $\alpha_n = \lfloor \log \log(n+2) \rfloor + 2$, $\beta_n = \lfloor \log n \rfloor + 2$, $s_n = \lfloor \log n \rfloor$, $t_n = n$, and $v_n = 2^n$. Then the conditions of Theorem 3.4 are satisfied.

Example 3.7. Fix some integer ℓ . Set $\alpha_n = \lfloor \log \log(n+2) \rfloor + 2$, $\beta_n = \lfloor \log n \rfloor + 2$, $s_n = \lfloor \log n \rfloor$, $t_n = \left\lfloor \left(\frac{\beta_n}{\alpha_n}\right)^{\ell+1} s_n \right\rfloor$, and $v_n = 2^n$. Then the conditions of Theorem 3.5 are satisfied.

Proof of Theorem 1.7. Let $\gamma = \dim_{\mathbb{M}}(S)$, $\alpha_i = 2$, $\beta_i = q_i$, $s_i = 0$, $t_i = 1$, $v_i = 1$, $F_i = 0$, and

$$I_i = S \cap \{0, \dots, q_i - 2\}.$$

Then (2.3) and (2.4) clearly hold. Note that

$$\mathcal{W}_Q(S) \subseteq \Theta(\alpha, \beta, s, t, v, F, I),$$

so $\dim_{\mathbb{H}}(\mathcal{W}_Q(S)) \leq \gamma$.

To get a lower bound, we construct a subset of $\mathcal{W}_Q(S)$ with Hausdorff dimension γ . To do this, let $T \subset \mathbb{N}$ be an infinite set that is sparse enough such that

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \chi_T(i) \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^k \log \#(S \cap \{0, \dots, q_i - 2\})} = 0.$$

Note that such a T exists since $\lim_{k \rightarrow \infty} \sum_{i=1}^k \log \#(S \cap \{0, \dots, q_i - 2\}) = \infty$.

Let $f : T \rightarrow S$ be a surjective function such that for all $t \in T$, we have $q_t > f(t)$. Such an f exists since $\min S < \min Q$, T is infinite, and Q is infinite in limit. Consider the homogeneous Moran set C with

$$n_k = \begin{cases} 1 & \text{if } k \in T \\ \#S \cap \{0, \dots, q_k - 2\} & \text{else} \end{cases}$$

and $c_k = \frac{1}{q_k}$ described as follows: If $k \in T$, then for any $x \in C$, $E_k(x) = f(k)$. Otherwise, $E_k(x) \in S \cap \{0, \dots, q_k - 2\}$. Since f is surjective, we have that for any $x \in C$ that $\mathcal{S}_Q(x) = S$, so $C \subseteq \mathcal{W}_Q(S)$. But

$$\begin{aligned} \dim_{\mathbb{H}}(C) &\geq \liminf_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_{k+1} n_{k+1}} \\ &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \chi_{\mathbb{N} \setminus T}(i) \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^k \log q_i + \log q_{k+1}} \\ &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^k \log q_i} \\ &= \lim_{k \rightarrow \infty} \frac{\log \#(S \cap \{0, \dots, q_k - 2\})}{\log q_k} = \gamma. \end{aligned}$$

Thus $\dim_{\mathbb{H}}(\mathcal{W}_Q(S)) \geq \gamma$, so we have $\dim_{\mathbb{H}}(\mathcal{W}_Q(S)) = \gamma$. □

4. FURTHER PROBLEMS

Problem 4.1. For which irrational x does there exist a basic sequence Q where $x \in \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$. The same question may be asked about several of the other sets discussed in this paper. We remark that it is already known that for every irrational x there exist uncountably many basic sequences Q where $x \in \mathcal{DN}(Q)$. See [9].

Problem 4.2. Prove that the conclusions of Theorem 3.4 and Theorem 3.5 hold for all Q that are infinite in limit and fully divergent.

Problem 4.3. In [12] sufficient conditions are given under countable intersections of sets of the form $\mathcal{DN}(Q) \setminus \bigcup_{j=1}^{\infty} \mathcal{RN}_j(Q)$ have full Hausdorff dimension. Surely a similar result holds for many of the sets described in this paper. Necessary and sufficient conditions similar to conditions found in the paper of W. M. Schmidt [17] may be possible.

REFERENCES

1. D. Airey, B. Mance, and J. Vandehey, *Normality preserving operations for Cantor series expansions and associated fractals part II*, arXiv:1407.0778.
2. M. T. Barlow and S. J. Taylor, *Defining fractal subsets of \mathbf{Z}^d* , Proc. London Math. Soc. (3) **64** (1992), no. 1, 125–152.
3. G. Cantor, *Über die einfachen Zahlensysteme*, Zeitschrift für Math. und Physik **14** (1869), 121–128.
4. P. Erdős and A. Rényi, *On Cantor's series with convergent $\sum 1/q_n$* , Annales Universitatis L. Eötvös de Budapest, Sect. Math. (1959), 93–109.
5. ———, *Some further statistical properties of the digits in Cantor's series*, Acta Math. Acad. Sci. Hungar **10** (1959), 21–29.
6. D. Feng, Z. Wen, and J. Wu, *Some dimensional results for homogeneous Moran sets*, Sci. China Ser. A **40** (1997), no. 5, 475–482.
7. J. Galambos, *Representations of real numbers by infinite series*, Lecture Notes in Math., vol. 502, Springer-Verlag, Berlin, Hiedelberg, New York, 1976.
8. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Dover, Mineola, NY, 2006.
9. P. Laffer, *Normal numbers with respect to Cantor series representation*, Ph.D. thesis, Washington State University, Pullman, Washington, 1974.
10. B. Li and B. Mance, *Number theoretic applications of a class of Cantor series fractal functions part II*, To appear in Int. J. Number Theory (2015).
11. B. Mance, *Number theoretic applications of a class of Cantor series fractal functions part I*, To appear in Acta Math. Hungar. (2014).
12. ———, *On the Hausdorff dimension of countable intersections of certain sets of normal numbers*, To appear in J. Théor. Nombres Bordeaux.
13. ———, *Typicality of normal numbers with respect to the Cantor series expansion*, New York J. Math. **17** (2011), 601–617.
14. A. Rényi, *On a new axiomatic theory of probability*, Acta Math. Acad. Sci. Hungar. **6** (1955), 329–332.
15. ———, *On the distribution of the digits in Cantor's series*, Mat. Lapok **7** (1956), 77–100.
16. ———, *Probabilistic methods in number theory*, Shuxue Jinzhan **4** (1958), 465–510.
17. W.M. Schmidt, *On normal numbers*, Pacific J. Math. **10** (1960), 661–672.
18. F. Schweiger, *Über den Satz von Borel-Rényi in der Theorie der Cantorschen Reihen*, Monatsh. Math. **74** (1969), 150–153.
19. P. Turán, *On the distribution of "digits" in Cantor systems*, Mat. Lapok **7** (1956), 71–76.
20. T. Šalát, *Cantorsche Entwicklungen der reellen Zahlen und das Hausdorffsche mass*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 15–41.
21. ———, *Über die Hausdorffsche Dimension der Menge der Zahlen mit beschränkten Folgen von Ziffern in Cantorschen Entwicklungen*, Czech. Math. J. **15 (90)** (1965), 540–553.
22. ———, *Über die Cantorschen Reihen*, Czech. Math. J. **18 (93)** (1968), 25–56.
23. ———, *Einige metrische Ergebnisse in der Theorie der Cantorschen Reihen und Bairesche Kategorien von Mengen*, Studia Sci. Math. Hungar. **6** (1971), 49–53.
24. H. Weyl, *Über ein Problem aus dem Gebiete der diophantischen Approximationen*, Nachr. Ges. Wiss. Göttingen, Math.-phys. **K1** (1914), 234–244.
25. ———, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352.

(D. Airey) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, 2515 SPEEDWAY, AUSTIN, TX 78712-1202, USA

E-mail address: dylan.airey@utexas.edu

(B. Mance) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, GENERAL ACADEMICS BUILDING 435, 1155 UNION CIRCLE, #311430, DENTON, TX 76203-5017, USA

E-mail address: mance@unt.edu